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Unstaggered Central Schemes for MHD and SMHD

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Abstract. We develop second-order nonoscillatory unstaggered central schemes (UCS) with a constrained transport-type method to solve one and two-dimensional hyperbolic problems arising in astrophysics. In contrast with the original central schemes that alternate the numerical solution on two staggered grids, the method we propose evolves the numerical solution on a single, but uses implicitly ghost staggered cells to bypass the resolution of the Riemann problems arising at the cell interfaces. To ensure an admissible physical solution when solving MHD/SMHD problems, we adapt the constrained transport method and apply it to our unstaggered central schemes.We numerically solve classical problems in astrophysics using the UCS method; the solenoidal property is satisfied at the discrete level thanks to the adapted constrained transport method and the obtained numerical results are in good agreement with their corresponding ones appearing in the recent literature, thus confirming the efficiency and potential of the scheme.

1. Introduction

The ideal MHD system consists of the conservation laws for the mass density ρ, momentum $\rho\mathbf{u}$, total energy ρe as well as Faraday's induction law:

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\mathbf{u}\\ \rho e\\ \mathbf{B}\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\mathbf{u}\\ \rho\mathbf{u}\mathbf{u}+I(p+\frac{\mathbf{B}\cdot\mathbf{B}}{2})-\mathbf{B}\mathbf{B}\\ (\rho e+p+\frac{\mathbf{B}\cdot\mathbf{B}}{2})\mathbf{u}-(\mathbf{u}\cdot\mathbf{B})\mathbf{B}\\ \mathbf{u}\mathbf{B}-\mathbf{B}\mathbf{u}\end{bmatrix}=0 \tag{1.1}$$

$\mathbf{B}$ is the magnetic field and I is the (3×3) identity matrix; the thermal pressure is computed from an ideal gas equation of state, $P=(\gamma-1)(\rho e-\frac{1}{2}\rho|\mathbf{u}|^2-\frac{1}{2}|\mathbf{u}|^2)$, where γ denotes the ratio of specific heats. The shallow water magnetohydrodynamic equations are obtained by integrating the three dimensional ideal MHD system in the vertical direction ($z-$direction) and by assuming that: 1) the mass density is constant, 2) the magnetohydrostatic pressure is constant at the surface, and 3) the equation for the magnetohydrostatic balance is satisfied: $\dfrac{\partial}{\partial z}(p+\frac{\rho}{2}|\mathbf{B}|^2)=-\rho g$.

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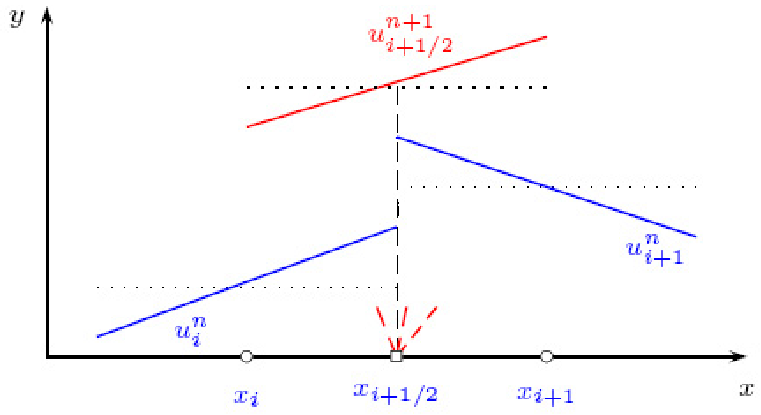

FIGURE 1. Geometry of the NT scheme

Under these condition the SMHD system becomes:

$$\frac{\partial}{\partial t}\begin{bmatrix} h \\ h\mathbf{u} \\ h\mathbf{B} \end{bmatrix} + \nabla \cdot \begin{bmatrix} h\mathbf{u} \\ h\mathbf{u}\mathbf{u} + \frac{gh^2}{2}\mathcal{I} - h\mathbf{B}\mathbf{B} \\ h\mathbf{u}\mathbf{B} - h\mathbf{B}\mathbf{u} \end{bmatrix} = \begin{bmatrix} 0 \\ -gh\nabla b \\ 0 \end{bmatrix}. \tag{1.2}$$

Entries of the matrix $\mathbf{Bu}$ in equations (1.1) and (1.2) are $(\mathbf{Bu})_{ij} = u_i B_j$.

2. One and two-dimensional unstaggered central schemes

Early in the nineties, Nessyahu and Tadmor (NT) presented a second-order accurate nonoscillatory central scheme for the approximate solution of hyperbolic systems [**10**]. The NT scheme is based on the staggered Lax-Friedrichs method; It avoids the resolution of the Riemann problems arising at the cell interfaces by evolving the numerical solution on an original grid and on a staggered dual one at consecutive time steps. To achieve second-order accuracy in space, the NT scheme evolves a piecewise linear numerical solution defined on the computational cells and uses slope-limiting to guarantee an oscillation-free numerical solution. However, the fact that the numerical solution, in the NT-type schemes, alternates between two grids at successive time steps is considered as a weakness of the method. More precisely, if the numerical solution obtained using an NT-type base scheme (at time t^n) requires additional treatment in order to satisfy a physical property, a synchronization problem arises since any treatment of the updated solution usually requires the solution values computed at different previous times (i.e., at time t^n, t^{n-1} and maybe earlier). The situation becomes even harder when the control cells of the original and the staggered grids are not of the same shape/type. If we assume that the solution u^n_i of the equation $u_t + f(u)_x = 0$ is known at time t^n on the cells $[x_{i-1/2}, x_{i+1/2}]$, then the NT scheme calculates the solution at time t^{n+1} at the centers of the staggered cells $[x_i, x_{i+1}]$ using the equation

$$u^{n+1}_{i+1/2} = \frac{1}{2}(u^n_i + u^n_{i+1}) + \frac{1}{8}\left[(u^n_i)' - (u^n_{i+1})'\right] - \frac{\Delta t}{\Delta x}\left[f(u^{n+1/2}_{i+1}) - f(u^{n+1/2}_i)\right] \tag{2.1}$$

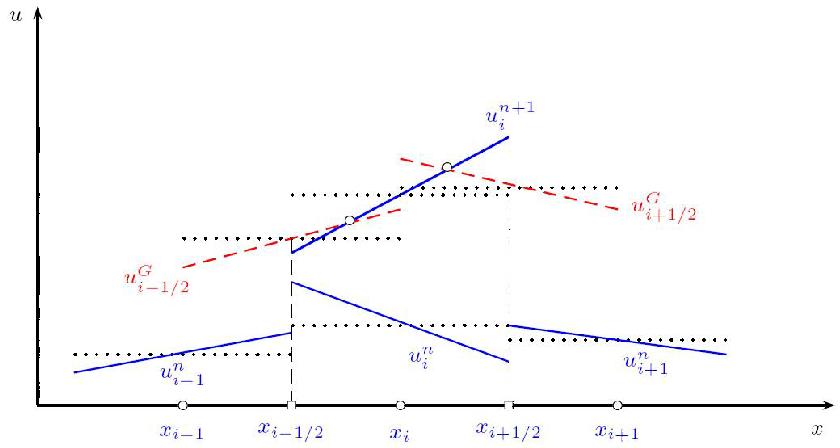

FIGURE 2. Geometry of the UCS method: The resolution of Riemann problems at cell interfaces is avoided thanks to the staggered ghost cells

where $(u_i^n)' \cong h\frac{\partial}{\partial x}u(x,t^n)|_{x=x_i} + O(\Delta x^2)$ approximates the slope to first-order accuracy; this leads to second-order spatial accuracy. Second-order temporal accuracy is obtained thanks to a predictor-corrector step. The solution on the original grid will be computed at time t^{n+2} using the equation

$$u_i^{n+2} = \frac{1}{2}(u_{i-1/2}^{n+1} + u_{i+1/2}^{n+1}) + \frac{1}{8}\left[(u_{i-1/2}^{n+1})' - (u_{i+1/2}^{n+1})'\right] - \frac{\Delta t}{\Delta x}\left[f(u_{i+1/2}^{n+3/2}) - f(u_{i-1/2}^{n+3/2})\right] \tag{2.2}$$

The complete description of the one-dimensional NT scheme is found in [**10**].

In 1998, Jiang et al. [**7**] presented a first unstaggered adaptation of the NT scheme; the method they proposed utilizes both iteration formulas of the original Nessyahu and Tadmor scheme (equations (2.1)-(2.2)), but with a fixed zero time-step ($\Delta t = 0$) in equation (2.2).
In a previous work [**15**] we have developed a one-dimensional unstaggered central scheme for the approximate solution of general hyperbolic systems; the method can be considered as an unstaggered adaptation of the NT scheme, and a generalization of the method presented by Jiang et al. [**7**]. The proposed unstaggered scheme evolves the numerical solution on a single grid and avoids the resolution of the Riemann problems arising at the cell interfaces, thanks to an implicitly used "ghost" staggered grid. The method summarizes as follows: Given the solution u_i^n on the computational grid, we obtain the solution $u_{i+1/2}^G$ at time t^{n+1} on the staggered ghost cells using equation (2.1). Since the numerical solution is defined using piecewise linear reconstruction of the piecewise constant solution at the cell centers, we define the solution u_i^{n+1} on the computational cells $[x_{i-1/2,i+1/2}]$ using the equation $u_i^{n+1} = \frac{1}{2}(u^G(x_{i-1/2} + \alpha\Delta x, t^{n+1}) + u^G(x_{i+1/2} - \alpha\Delta x, t^{n+1})$, where $0 < \alpha < 0.5$. Using linear interpolants, the solution at time t^{n+1} becomes $u_i^{n+1} = \frac{1}{2}(u_{i-1/2}^G + u_{i+1/2}^G) + \frac{\alpha}{2}((u_{i-1/2}^G)' - (u_{i+1/2}^G)')$. If we set $\alpha = 1/4$ we obtain

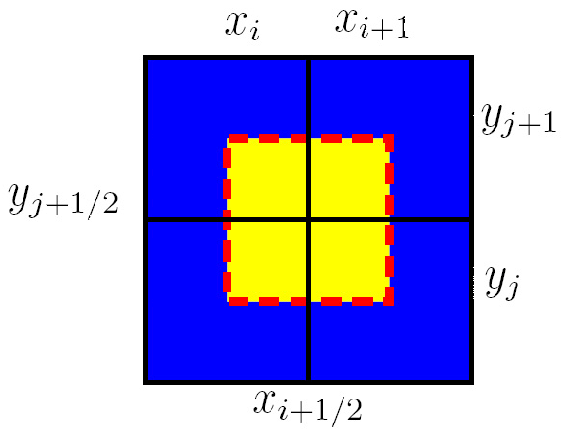

FIGURE 3. Geometry of the two-dimensional staggered central schemes.

the formulas proposed by Jiang et al. [**7**], i.e., the second iteration formula of the original NT scheme computed with $\Delta t = 0$.

Two-dimensional extension of the original one-dimensional Nessyahu and Tadmor scheme were developed in [**1, 8**]; as it is the case with the original NT scheme, the two-dimensional extension evolves the numerical solution on an original grid and a staggered dual one. Cells of both original and dual staggered grids are Cartesian cells. Figure 3 shows four cells ($C_{i,j}$, $C_{i+1,j}$, $C_{i,j+1}$, and $C_{i+1,j+1}$) of the original grid and the dual cell $D_{i+1/2,j+1/2}$ of the staggered grid. If the numerical solution u^n_{ij} of the equation $u_t + f(u)_x + g(u)_y = 0$ is computed on the original grid at time t^n, then the solution at time t^{n+1} will be computed on the staggered grid using the formula:

$$\begin{aligned} u^{n+1}_{i+1/2,j+1/2} &= \frac{1}{4}(u^n_{i+1,j} + u^n_{i+1,j+1} + u^n_{i,j+1} + u^n_{i,j+1}) \\ &+ \frac{1}{16}(u^{lim}_{i,j;x} - u^{lim}_{i+1,j;x}) - \frac{\lambda}{2}\left[f^{n+1/2}_{i+1,j} - f^{n+1/2}_{i,j}\right] \\ &+ \frac{1}{16}(u^{lim}_{i,j+1;x} - u^{lim}_{i+1,j+1;x}) - \frac{\lambda}{2}\left[f^{n+1/2}_{i+1,j+1} - f^{n+1/2}_{i,j+1}\right] \\ &+ \frac{1}{16}(u^{lim}_{i,j;y} - u^{lim}_{i,j+1;x}) - \frac{\lambda}{2}\left[g^{n+1/2}_{i,j+1} - g^{n+1/2}_{i,j}\right] \\ &+ \frac{1}{16}(u^{lim}_{i+1,j;y} - u^{lim}_{i+1,j+1;x}) - \frac{\lambda}{2}\left[g^{n+1/2}_{i+1,j+1} - g^{n+1/2}_{i+1,j}\right] \end{aligned} \tag{2.3}$$

where $\lambda = \Delta t/\Delta x$, and $(u^{lim}_x/\Delta x, u^{lim}_y/\Delta y)$ is a limited gradient of the numerical solution. A similar formula computes the numerical solution on the cells of the original grid at time t^{n+2}. A detailed description of the two-dimensional extension of the NT scheme is found in [**1, 8**].

The two-dimensional version of the UCS method evolves a piecewise linear numerical solution on a unique grid where the nodes x_{ij} are the centers of the cells $C_{ij} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$. If the numerical solution u^n_{ij} is known at time t^n, we obtain the solution u^{n+1}_{ij} at the following time $t^{n+1} = t^n + \Delta t$ as follows: First we obtain an update of the solution on the ghost cells $G_{i+1/2,j+1/2} =$

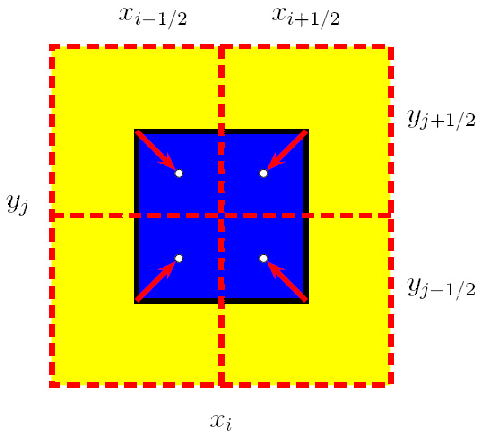

FIGURE 4. Geometry of the two-dimensional unstaggered central schemes; linear interpolations of the solution on the ghost cells define the numerical solution on a single grid while avoiding the time-consuming resolution of the Riemann problems at the cell interfaces.

$[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ using equation (2.3). As in the one-dimensional case, the solution at the center of the cell C_{ij} is obtained using a piecewise linear reconstruction of the piecewise constant solution defined at the centers of the cells $G_{i+1/2,j+1/2}$ (figure 4) using the formula:

(2.4)
$$u_{i,j}^{n+1} = \frac{1}{4}\Big[u^G(x_{i-1/2} + \alpha\Delta x, y_{j-1/2} + \beta\Delta y) + u^G(x_{i+1/2} - \alpha\Delta x, y_{j-1/2} + \beta\Delta y) + u^G(x_{i+1/2} - \alpha\Delta x, y_{j+1/2} - \beta\Delta y) + u^G(x_{i-1/2} + \alpha\Delta x, y_{j+1/2} - \beta\Delta y)\Big],$$

where the interpolated values in equation (2.4) are obtained as follows:

(2.5)
$$u^G(x_{i-1/2} \pm \alpha\Delta x, y_{j-1/2} \pm \beta\Delta y) = u^G_{i-1/2,j-1/2} \pm \alpha u^{G,lim}_{i-1/2,j-1/2;x} \pm \beta u^{G,lim}_{i-1/2,j-1/2;y}.$$

Again $(\nabla \mathrm{u}^G) \equiv (u_x^{G,lim}/\Delta x + \mathcal{O}(\Delta x), u_y^{G,lim}/\Delta y + \mathcal{O}(\Delta y))$ is a limited gradient of the numerical solution calculated on the ghost cells. The parameters α and β in equation (2.5) range between 0 and 1/2. Here again for $\alpha = \beta = 1/4$ we obtain the same iteration formula as in Jiang et al. [**7**]. We note that the one and two-dimensional UCS methods have the same stability condition as the original central schemes presented in [**10**] and [**1, 8**], respectively.

3. Treatment of the nonsolenoidal magnetic field/flux

The accumulation of numerical errors such as the truncation and round-off errors usually leads to a numerical solution that does not satisfy the constraint $\nabla \cdot \mathbf{B} = 0$ in the case of the MHD equations and the constraint $\nabla \cdot (h\mathbf{B}) = 0$ in the

case of the SMHD equations. This means the formation of magnetic monopoles, a phenomena that has never been observed neither in nature nor in laboratory. Among many useful methods to ensure a physically admissible numerical solution, Evans and Hawley's Constrained Transport (CT) approach [**6**] has proven to be very efficient, and at the same time very simple to use. In a previous work [**14**] we have constructed special CT-type methods that apply to the case of staggered NT-type central schemes with Cartesian or diamond-shaped dual cells in two and three space dimensions. In this section we adapt the CT approach to the case of UCS methods. Let $\mathcal{B}$ denote the magnetic field $\mathbf{B}$ in the case of MHD equations or the magnetic flux $h\mathbf{B}$ in the case of the SMHD equations. Assume that u^n_{ij} denote the solution at time t^n defined at the center x_{ij} of the cells C_{ij}, and assume that the constraint $\nabla\cdot\mathcal{B}^n_{ij}=0$ is satisfied, i.e., the central difference discretization of the divergence operator satisfies the equation

$$\nabla\cdot\mathcal{B}^n_{ij}\approx\frac{\mathcal{B}^n_{1;i+1,j}-\mathcal{B}^n_{1;i-1,j}}{2\Delta x}+\frac{\mathcal{B}^n_{2;i,j+1}-\mathcal{B}^n_{2;i,j-1}}{2\Delta y}=0. \tag{3.1}$$

We compute the numerical solution u^{n+1}_{ij} at time t^{n+1} using the UCS base scheme; the magnetic field/magnetic flux $\mathcal{B}$ in the updated solution at time t^{n+1} is not solenoidal in general, and needs to be corrected. First we compute the $z-$component $\Omega=-u_1\mathcal{B}_2+u_2\mathcal{B}_1$ of the electric field $\mathbf{E}$ at time $t^{n+1/2}$ using the numerical solution obtained at times t^n and t^{n+1} on the original and ghost grid, as follows:

$$\Omega^{n+1/2}_{i+1/2,j+1/2}=\frac{1}{2}\left[\Omega^{n+1}_{i+1/2,j+1/2}+\frac{\Omega^n_{i,j}+\Omega^n_{i+1,j}+\Omega^n_{i+1,j+1}+\Omega^n_{i,j+1}}{4}\right] \tag{3.2}$$

We note that this special discretization of the electric field preserves the temporal second-order of accuracy of the base scheme.
Next, we discretize the magnetic field/flux induction equation

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathcal{B}_1\\ \mathcal{B}_2\end{pmatrix}-\frac{\partial}{\partial x}\begin{pmatrix}0\\ \Omega\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}\Omega\\ 0\end{pmatrix}=0$$

using central differences, and update the magnetic field/flux on the ghost cells as follows:

$$(\mathcal{B}_1)^{n+1}_{i+1/2,j+1/2}=\frac{(\mathcal{B}_1)^n_{i,j}+(\mathcal{B}_1)^n_{i+1,j}+(\mathcal{B}_1)^n_{i+1,j+1}+(\mathcal{B}_1)^n_{i,j+1}}{4}-\Delta t\frac{\Omega^{n+1/2}_{i+1/2,j+3/2}-\Omega^{n+1/2}_{i+1/2,j-1/2}}{2\Delta y} \tag{3.3}$$

$$(\mathcal{B}_2)^{n+1}_{i+1/2,j+1/2}=\frac{(\mathcal{B}_2)^n_{i,j}+(\mathcal{B}_2)^n_{i+1,j}+(\mathcal{B}_2)^n_{i+1,j+1}+(\mathcal{B}_2)^n_{i,j+1}}{4}+\Delta t\frac{\Omega^{n+1/2}_{i+3/2,j+1/2}-\Omega^{n+1/2}_{i-1/2,j+1/2}}{2\Delta x} \tag{3.4}$$

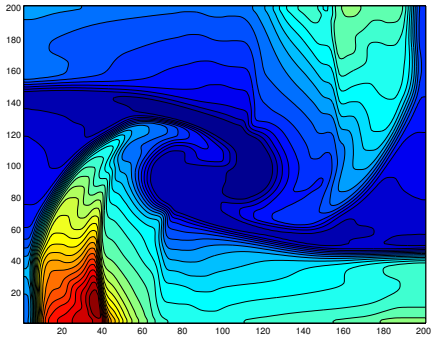

FIGURE 5. Mass density contours for the 2D MHD Riemann problem at time t=0.8.

It is easily shown that with this special symmetric discretization of the induction equation, the magnetic field/flux on the ghost cells is divergence-free and we have:

$$\nabla\cdot(\mathcal{B})^{n+1}_{i+1/2,j+1/2} = \frac{1}{4}\big[\nabla\cdot(\mathcal{B})^{n}_{i,j} + \nabla\cdot(\mathcal{B})^{n}_{i+1,j} + \nabla\cdot(\mathcal{B})^{n}_{i+1,j+1} + \nabla\cdot(\mathcal{B})^{n}_{i,j+1}\big] \equiv 0.$$

This means that if $\mathcal{B}$ in the initial condition is solenoidal, then it will remain as such at the following time step on the ghost cells. The magnetic feild/flux on the original computational grid is finally obtained as:

$$(\mathcal{B})^{n+1}_{ij} = \frac{1}{4}\Big[(\mathcal{B})^{n+1}_{i-1/2,j-1/2} + (\mathcal{B})^{n+1}_{i+1/2,j-1/2} + (\mathcal{B})^{n+1}_{i+1/2,j+1/2} + (\mathcal{B})^{n+1}_{i-1/2,j+1/2}\Big].$$

4. Numerical Experiments

In this section we apply the UCS method and solve classical problems arising in MHD and SMHD. In both cases the magnetic field/flux is systematically treated using the constrained transport method.

4.1. MHD 2D Riemann problem. First we consider the two-dimensional MHD Riemann problem as considered previously in [**5**]: The initial conditions for (ρ, p, u_x, u_y) are given by $(1, 1, 0.75, 0.5)$ if $x > 0$ and $y > 0$, (2,1,0.75,0.5) if $x < 0$ and $y > 0$, (1,1,-0.75,0.5) if $x < 0$ and $y < 0$ and (3,1,-0.75,-0.5) if $x > 0$ and $y < 0$. The initial magnetic field $\mathbf{B} = (2, 0, 1)$ is uniform in the rectangle $[-1, 1]^2$. The numerical solution is computed at time $t = 0.8$ on a 200×200 grid using the MC-θ ($\theta = 1.5$) limiter.

Figure 5 shows the contour lines for the mass density while figure 6 shows the divergence of the magnetic field. These results are in good agreement with the results appearing in [**5**]. Thanks to the constrained transport divergence treatment, the maximum absolute value of the divergence of the magnetic field observed for this problem is $7.16093e - 013$.

4.2. SMHD Blast wave problem. For our next experiment, we consider the two-dimensional SMHD explosion problem as presented in [**9**]; the initial condition for this problem for $[h, u, v, Bx, By]$ is $[1, 0, 0, 0.1, 0]$ if $||x|| < 0.3$ and $[0.1, 0, 0, 1, 0]$

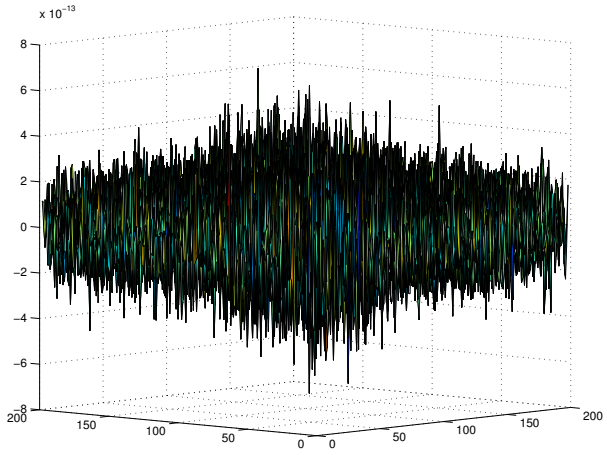

FIGURE 6. Mass density contours for at $\nabla \cdot \mathbf{B}$ of the 2D MHD Riemann problem at the final time.

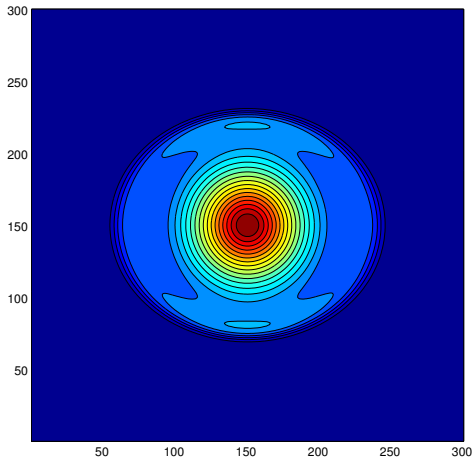

FIGURE 7. Mass density contours for the 2D SMHD explosion problem at time t=0.25 (left)

if $||x|| > 0.3$ in the computational domain is $[-1,1]^2$. The numerical solution is computed at time $t = 0.25$ on a 300×300 grid. Figure 7 shows the mass density contours propagating outward radially; Figure 8 shows the divergence of the magnetic flux to the right. Figure 8 shows that the maximum absolute value of the divergence of the magnetic flux for this problem remains within a $10e-14$ threshold. The obtained numerical results compare very well with their corresponding ones in [**9**], thus confirming the potential and efficiency of the method

5. Conclusion

We have presented an unstaggered central finite volume method for the approximate solution of general hyperbolic problems in one and two space dimensions.

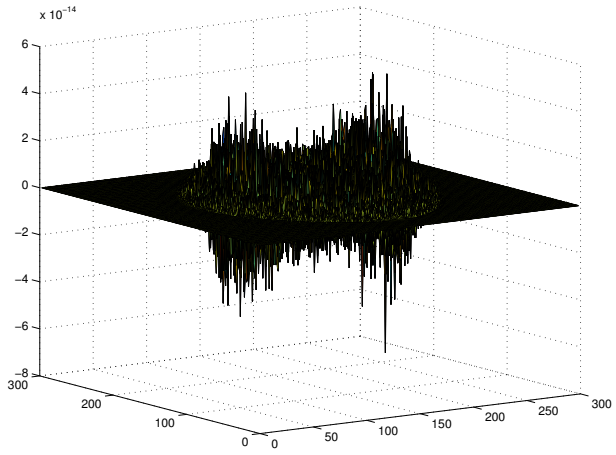

FIGURE 8. $\nabla(h\mathbf{B})$ of the 2D SMHD explosion problem at the final time. (right).

The UCS method is second-order accurate thanks to piecewise linear interpolants, and avoids the resolution of the Riemann problems at the cell interfaces thanks to a ghost grid of staggered cells implicitly used. To maintain a divergence-free magnetic field/flux in the numerical solution of MHD/SMHD, problems we have adapted Evans and Hawley's Constrained Transport method to the UCS method and successfully solved classical problems. In both cases the obtained numerical results are in good agreement with their corresponding ones appearing in the recent literature and the divergence-free property of the magnetic field/flux is satisfied at the discrete level, thus confirming the potential and the efficiency of the proposed method.

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